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An Integrodifferential Model for Phase Transitions: Stationary Solutions in Higher Space Dimensions

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We study the existence and stability of stationary solutions of an integrodifferential model for phase transitions, which is a gradient flow for a free energy functional with general nonlocal integrals penalizing spatial nonuniformity. As such, this model is a nonlocal extension of the Allen–Cahn equation, which incorporates long-range interactions. We find that the set of stationary solutions for this model is much larger than that of the Allen–Cahn equation.

KEY WORDS: Nonlocal Allen–Cahn equation; long-range interaction; pinning.

1. INTRODUCTION

We study the integral equation

$$(J * u)(x) - u(x) - \lambda f(u(x)) = 0, \qquad x \in R^n, \quad n \geqslant 1 \tag{1.1}$$

where $J * u(x)(=\int_{R^n} J(x-y)\, u(y)\, dy)$ is the convolution of J and u and $\lambda > 0$. We assume $\int_{R^n} J(x)\, dx = 1$ and f is bistable, e.g., $f(u) = u(u^2 - 1)$. Solutions to (1.1) are stationary solutions of the evolution equation

$$u_t = J * u - u - \lambda f(u) \tag{1.2}$$

which may be thought of as a nonlocal version of the Allen–Cahn equation.[(1)]

We find that, in the special case where $u + \lambda f(u)$ is nonmonotone and λ is sufficiently large, there exist stationary solutions having discontinuities across arbitrarily prescribed interfaces. We construct both stable and unstable solutions of this type. An important point is that some of our results allow for J to take negative as well as positive values.

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Equation (1.2), recently proposed in ref. 3, can model a variety of physical and biological phenomena, e.g., a material whose state is described by an order parameter. Note that (1.2) is an L^2-gradient flow of the free energy functional

$$E(u)=\frac{1}{4}\int_{R^n}\int_{R^n} j(x-y)(u(x)-u(y))^2\,dx\,dy+\int_{R^n} W(u(x))\,dx \tag{1.3}$$

with $J=j/\int_{R^n} j$, $f(u)=W'(u)$ and $\lambda=1/\int_{R^n} j$. Here, W is a double-well function with two (not necessarily equal) local minima (say, at -1 and 1), and $j(r)$ is a measure of the energy resulting from $u(x)$ being different from $u(x+r)$. The first term in (1.3) penalizes spatially inhomogeneous materials when $j>0$, and the second term (bulk term) penalizes states which take values other than the two minima of W.

From a microscopic viewpoint, one can obtain (1.3) from the following heuristic argument, which could be justified in the mean field approximation with a careful analysis using multiple scalings.

Consider a binary alloy, whose lattice sites are occupied by blocks, each block consisting of many atoms of species A and B. To a site with a block of A (B) atoms only, we assign the spin $+1$ (-1). Let $v(r)$ be the probability that the spin at site r is $+1$. Then the expected value of this spin is $u(r)=2v-1\in[-1,1]$.

The Helmholtz free energy of our system is given by

$$E=H-TS \tag{1.4}$$

where H is the internal interaction energy, T the absolute temperature and S the entropy.

Similar to the Ising model, the internal energy, in the mean field approximation, has the form

$$H(u)=-\frac{1}{4}\sum_{r,r'}\big[J^{AA}_{rr'}(1+u(r))(1+u(r'))+J^{BB}_{rr'}(1-u(r))(1-u(r'))$$
$$+J^{AB}_{rr'}((1+u(r))(1-u(r'))+(1-u(r))(1+u(r')))\big]$$

where $J^{AA}_{rr'}$, $J^{BB}_{rr'}$, $J^{AB}_{rr'}=J^{BA}_{rr'}$ denote the interaction energies between sites r and r' with spins 1 and 1, -1 and -1, 1 and -1 (ref. 32), respectively, which we assume are positive, translation invariant, and symmetric in (r,r').

In the Bragg–Williams[10] approximation, the total entropy has the form

$$S(u)=-K\sum_r\big[(1+u(r))\log(1+u(r))+(1-u(r))\log(1-u(r))\big]$$

where $K > 0$. Rearranging (1.4), we get

$$E(u) = \frac{1}{4} \sum_{r, r'} \left(\tfrac{1}{2} J^{AA}_{rr'} + \tfrac{1}{2} J^{BB}_{rr'} - J^{AB}_{rr'}\right)(u(r) - u(r'))^2 - \tfrac{1}{2} k_1 + \sum_{r} \Big(-\tfrac{1}{2} k_2(r)\, u(r) + TK[(1 + u(r)) \log(1 + u(r)) + (1 - u(r)) \log(1 - u(r))] - \tfrac{1}{2} k_3(r)\, u^2(r)\Big) \tag{1.5}$$

where

$$k_1 = \sum_{r, r'} \left(\tfrac{1}{2} J^{AA}_{rr'} + \tfrac{1}{2} J^{BB}_{rr'} + J^{AB}_{rr'}\right)$$

$$k_2(r) = \sum_{r'} \left(J^{AA}_{rr'} - J^{BB}_{rr'}\right)$$

$$k_3(r) = \sum_{r'} \left(\tfrac{1}{2} J^{AA}_{rr'} + \tfrac{1}{2} J^{BB}_{rr'} - J^{AB}_{rr'}\right)$$

We assume that $k_3(r) > 0$. Note that for TK and $|k_2|$ small enough, the last summand in (1.5) has two minima, which are of equal depth if $k_2(r) = 0$. We replace this summand by $\sum_r W(u(r), r)$ and assume W is a double-well function everywhere defined and with minima at -1 and 1. The constant k_1 can be dropped since it does not effect the dynamics or the location of critical points. Assuming that $J_{rr'} \equiv \frac{1}{2} J^{AA}_{rr'} + \frac{1}{2} J^{BB}_{rr'} - J^{AB}_{rr'}$ and $W(u, r)$ are spatially homogeneous (i.e., $J_{rr'} = j(r - r')$ and $W(u, r) = W(u)$) and taking the continuum limit in (1.5) then gives (1.3). Note that this derivation does not require j to be everywhere positive.

In what follows, we often require $g(u) \equiv u + \lambda f(u)$ to be nonmonotone. This will not be the case without a modification to the above derivation of (1.2) and (1.3). Observe that the L^2-gradient of E is

$$-j * u + \left(\int j\right) u + W'(u)$$

where

$$W'(u) = -\tfrac{1}{2} k_2 - TS'(u) - \left(\int j\right) u$$

so

$$u + \lambda f(u) = -\lambda\left(\tfrac{1}{2} k_2 + TS'(u)\right)$$

which is monotone. One possibility is to incorporate a short-range interaction by having $j = j_l + j_s$ with $j_s = c\delta$, where δ is the Dirac delta function and $c > 0$. (Note that, if $j_s(x) = m^n \bar{j}(mx)$ for some positive, integrable function $\bar{j}$ and we take the limit as $m \to \infty$, then $j_s \to c\delta$ where $c = \int \bar{j}$). With this, the gradient of E becomes

$$
\begin{aligned}
-j * u + & \left(\int j \right) u + W'(u) \\
& = -j_l * u(x) - cu(x) + \left(\int j_l + c \right) u(x) + W'(u(x)) \\
& = -j_l * u + \left(\int j_l \right) u + W'(u)
\end{aligned}
$$

We take $J = j_l / \int_{R^n} j_l$ and $\lambda = 1/\int_{R^n} j_l$ so that now

$$u + \lambda f(u) = -\lambda(\tfrac{1}{2}k_2 + TS'(u) + cu)$$

which is nonmonotone for small values of T. From now on we will just assume that g is nonmonotone and that $j_l = j$.

Functional (1.3) is a natural generalization of the well-known and studied Ginzburg–Landau functional

$$E^l(u) = \frac{1}{2} \int_{R^n} |\nabla u(x)|^2 \, dx + \int_{R^n} W(u(x)) \, dx \tag{1.6}$$

Namely, we change variables in the first integral of (1.3) using $\eta = (x - y)/2$, $\xi = (x + y)/2$ and then expand $u(x) = u(\xi + \eta)$ and $u(y) = u(\xi - \eta)$ about ξ, to get the formal expression

$$2^n \int_{R^n} \int_{R^n} j(2\eta) \left(\sum_{k=1}^{\infty} \sum_{|\alpha| = 2k-1} \frac{1}{\alpha!} \partial^\alpha u(\xi) \, \eta^\alpha \right)^2 d\xi \, d\eta \tag{1.7}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and $\eta^\alpha = \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}$.

Truncating the summation in (1.7) at $k = 1$ gives, for some $c > 0$, an energy

$$\int_{R^n} [c \, |\nabla u|^2 + W(u)] \, dx$$

which is the same as $E^l(u)$ up to the constant c which can be absorbed by a further change of variable. Thus $E^l(u)$ can be treated as a first order approximation of $E(u)$. Consequently, the Allen–Cahn equation[1]

$$u_t = \Delta u - \lambda f(u)$$

can be regarded as the first order approximation of (1.2).

Other integrodifferential equations with many of the properties of (1.2) have been studied in refs. 27, 14–17, 24–26, 31 (continuum limits of dynamic Ising models), refs. 6–9, 28–30 (nonlocal ferromagnetism), ref. 18 (neural networks) and in refs. 19 and 20 (nonlocal elasticity). In particular, nonlocal versions of the Cahn–Hilliard equation[11] have been studied in refs. 22 and 23.

Monotone traveling and stationary waves for (1.2) with $n = 1$ were studied in ref. 3, where discontinuous waves were also discovered. Subsequently, Chen in ref. 12 found some nonmonotone discontinuous waves. Stationary homoclinic solutions were constructed by Chmaj and Ren in ref. 13. In all these, it was observed that pinning may occur when a discontinuous solution exists, even though the potential wells are of unequal depth.

In ref. 2 we study the discrete version of this equation.

Interfacial motion in higher space dimensions was studied in refs. 15, 21, 24–26, and 31.

Stationary solutions of a higher order equation derived from (1.7) were studied in ref. 5.

The paper is organized as follows: In Section 2 we study the case where $j \geqslant 0$. In Section 3 we extend our results to the case where j changes sign and λ is large. In Section 4, we study global stability for $n = 1$ and λ large. In Section 5, we prove a result giving the asymptotic behavior of solutions to (1.1). In particular, solutions do not in general approach their limits exponentially at $\pm\infty$.

2. THE CASE $j \geqslant 0$

We assume that $f \in C^r$, $r \geqslant 2$, f has three zeros -1, a and 1, such that $f'(-1) > 0$, $f'(a) < 0$ and $f'(1) > 0$ and $g(u) \equiv u + \lambda f(u)$ has three zeros and exactly three intervals of monotonicity. About J (recall that $J = j/\int_{R^n} j$) we assume that $J \geqslant 0$ and $J \in W^{k,1}(R^n)$, $k \geqslant 1$. Let $m \equiv \min\{r, k\}$.

Let $u_\lambda^- < u_\lambda^+$ be the extreme roots of $u + \lambda f(u) = 0$. Note that, $u_\lambda^- > -1$, $u_\lambda^+ < 1$ and $u_\lambda^\pm \to \pm 1$ as $\lambda \to \infty$.

Let $\alpha^{\pm}$, δ be such that $u_\lambda^- < \alpha^- < a < \alpha^+ < u_\lambda^+$,

$$f'(u) \geqslant \delta > 0 \qquad \text{for} \quad u \in [-1, \alpha^-] \cup [\alpha^+, 1] \tag{2.1}$$

$$\lambda f(\alpha^-) \geqslant (1 - \alpha^-) \sup_{x \in M^c} \int_M J(x-y)\, dy \tag{2.2}$$

$$-\lambda f(\alpha^+) \geqslant (1 + \alpha^+) \sup_{x \in M} \int_{M^c} J(x-y)\, dy \tag{2.3}$$

where M is a given measurable set and M^c is its complement.

Theorem 2.1. Assume that f satisfies the hypotheses at the beginning of this section. For any $\lambda > 0$ and measurable set M satisfying (2.1), (2.2) and (2.3) there exists a unique solution $\hat{u}$ to (1.1), such that

$$\hat{u}(x) \begin{cases} \geqslant \alpha^+ & \text{for} \quad x \in M, \\ \leqslant \alpha^- & \text{for} \quad x \in M^c \end{cases}$$

Moreover, $\hat{u}$ is C^0 on M and M^c, C^m on int M and int M^c and (locally) asymptotically stable in the $L^\infty(R^n)$ norm.

Proof. First we prove existence. Let

$$B = \{u \in L^\infty(R^n) : u(x) \in [\alpha^+, 1] \text{ for } x \in M \text{ and } u(x) \in [-1, \alpha^-] \text{ for } x \in M^c\}$$

Define $T: B \to L^\infty(R^n)$ by

$$Tu(x) = u(x) + h[J * u(x) - u(x) - \lambda f(u(x))]$$

We show that for $h > 0$ small enough T is a contraction mapping.

Since $|J * u(x)| \leqslant \|u\|_\infty$, taking h so small that

$$h\lambda f'(u) < 1 - h \qquad \text{for} \quad u \in [-1, \alpha^-] \cup [\alpha^+, 1]$$

ensures that $-1 \leqslant Tu(x) \leqslant 1$ for all $u \in B$. Furthermore, since $u - h[u + \lambda f(u)]$ is increasing on $[\alpha^+, 1]$, for $x \in M$,

$$Tu(x) \geqslant \alpha^+ + h\left[\alpha^+ \int_M J(x-y)\, dy - \int_{M^c} J(x-y)\, dy - \alpha^+ - \lambda f(\alpha^+)\right]$$

$$= \alpha^+ - h\left[(1 + \alpha^+) \int_{M^c} J(x-y)\, dy + \lambda f(\alpha^+)\right] \geqslant \alpha^+$$

where we used (2.3) in the last inequality.

Similarly, for $x \in M^c$, (2.2) implies that $Tu(x) \leqslant \alpha^-$, therefore

$$T: B \to B \tag{2.4}$$

Let $u, v \in B$, then

$$\begin{aligned} \|Tu - Tv\|_\infty &\leqslant (1 - h(1 + \delta\lambda))\ \|u - v\|_\infty + h\ \|J*(u-v)\|_\infty \\ &\leqslant (1 - h\delta\lambda)\ \|u - v\|_\infty \end{aligned}$$

provided $h(1 + \lambda f'(u)) < 1$ for $u \in [-1, \alpha^-] \cup [\alpha^+, 1]$. Thus T is a contraction for

$$0 < h < \frac{1}{1 + \lambda \max_{u \in [-1, \alpha^-] \cup [\alpha^+, 1]} f'(u)}$$

and has a unique fixed point $\hat{u} \in B$. Clearly, $\hat{u}$ is our desired solution.

On each of the sets M and M^c, (1.1) can be rewritten as

$$\hat{u} = g_i^{-1}(J * \hat{u}), \qquad i = 1, 2$$

where $g(u) = u + \lambda f(u)$ and g_i^{-1} is defined to be one of the two extreme branches of g^{-1}. Since $J * \hat{u}$ is C^k, we conclude that $\hat{u}$ is C^0 on M and M^c, C^m on int M and int M^c.

Remark 2.2. Note that if λ is large enough, (2.2) and (2.3) do not place any constraint on M.

We now show that our solution $\hat{u}$ is asymptotically, exponentially stable.

For some positive ε and β to be chosen later, define

$$\bar{u}(x, t) = \hat{u}(x) + \varepsilon e^{-\beta t}$$

Let $Nv \equiv v_t - (J * v - v - \lambda f(v))$. It is easily seen that

$$\begin{aligned} N\bar{u} &= -\beta\varepsilon e^{-\beta t} - (J * \hat{u} - \hat{u} - \lambda f(\hat{u} + \varepsilon e^{-\beta t})) \\ &= -\beta\varepsilon e^{-\beta t} + \lambda f(\hat{u} + \varepsilon e^{-\beta t}) - \lambda f(\hat{u}) \end{aligned}$$

By Taylor's expansion, $|\lambda f(\hat{u} + \varepsilon e^{-\beta t}) - \lambda f(\hat{u}) - \lambda f'(\hat{u})\, \varepsilon e^{-\beta t}| \leqslant C(\varepsilon e^{-\beta t})^2$ for some $C > 0$. Thus

$$N\bar{u} \geqslant -\beta\varepsilon e^{-\beta t} + \lambda f'(\hat{u})\, \varepsilon e^{-\beta t} - C(\varepsilon e^{-\beta t})^2$$

By (2.1), we can choose positive ε and β small enough such that $N\bar{u}(x, t) \geqslant 0$ for all $t > 0$, thus $\bar{u}$ is a supersolution for all $t > 0$.

In a similar way,

$$\underline{u}(x,t) \equiv \hat{u}(x) - \varepsilon e^{-\beta t}$$

is a subsolution for ε and β chosen as before and all $t > 0$.

Consider the Cauchy problem: (1.2) with initial data u_0, such that

$$\hat{u}(x) - \varepsilon \leqslant u_0(x) \leqslant \hat{u}(x) + \varepsilon$$

The comparison principle for (1.2) implies that the solution $u(x,t)$ of the Cauchy problem satisfies

$$\hat{u}(x) \leqslant \liminf_{t \to \infty} u(x,t) \leqslant \limsup_{t \to \infty} u(x,t) \leqslant \hat{u}(x)$$

thus $\hat{u}$ is exponentially, asymptotically stable in $L^\infty(R^n)$ norm.

Remark 2.3. In the above proof, we used the following comparison principle:

Let $u, v \in L^\infty(R \times [0,T])$ satisfy

$$u_t - (J * u - u) + f(u) \geqslant v_t - (J * v - v) + f(v) \text{ on } R \times [0,T]$$

and $-1 \leqslant v(x,0) \leqslant u(x,0) \leqslant 1$. Then $-1 \leqslant v(x,t) \leqslant u(x,t) \leqslant 1$.

This can be proved (under the assumption $\int_R y^2 J(y)\, dy < \infty$) by a modification of an arguyment on p. 79 in ref. 31.

3. THE CASE λ LARGE

Let f and J be as in Section 2, but now we allow J to change sign. Note that the existence proof from Section 2 cannot be used here, since in this case T defined in (2.4) in general does not map B into B.

We proceed as follows. First, we establish *a priori* estimates for $L^\infty(R^n)$ solutions of (1.1). Then with the help of a continuation argument, for λ large enough, we construct all stationary solutions to (1.1).

We assume that f has at least linear growth outside $[-1,1]$. Let

$$\begin{cases} \lambda f(u) \leqslant d(u+1) & \text{for } \quad u \leqslant -1, \\ \lambda f(u) \geqslant d(u-1) & \text{for } \quad u \geqslant 1 \end{cases} \tag{3.1}$$

for some $d > 0$. Define

$$P \equiv \{x \in R^n : J(x) > 0\},$$
$$N \equiv \{x \in R^n : J(x) < 0\}$$

Assume that

$$d > 2 \int_N |J(x)|\, dx \tag{3.2}$$

Proposition 3.1. Assume (3.2) holds. Then any $L^\infty(R^n)$ solution u to (1.1) satisfies

$$\frac{-d}{d-2\int_N |J|} \leqslant u(x) \leqslant \frac{d}{d-2\int_N |J|} \qquad \text{for all} \quad x \in R^n \tag{3.3}$$

Proof. Set $M \equiv \sup_{x\in R^n} u(x)$, $m \equiv \inf_{x\in R^n} u(x)$. Let $\{x_n^M\}_n$ be a sequence such that $u(x_n^M) \to M$ as $n \to \infty$, and $\{x_n^m\}_n$ a sequence such that $u(x_n^m) \to m$ as $n \to \infty$. Then

$$\begin{aligned}
\lambda f(u(x_n^M)) - M \int_P J(y)\,dy - m \int_N J(y)\,dy \\
= \int_P J(y)(u(x_n^M + y) - M)\,dy \\
+ \int_N J(y)(u(x_n^M + y) - m)\,dy - u(x_n^M) \leqslant -u(x_n^M)
\end{aligned}$$

Passing to the limit $n \to \infty$, we get

$$\lambda f(M) \leqslant (M - m) \int_N |J| \tag{3.4}$$

A similar argument shows that

$$\lambda f(m) \geqslant (m - M) \int_N |J| \tag{3.5}$$

Now, if $m \geqslant -1$ and $M \leqslant 1$ then obviously (3.3) is satisfied. Let us assume that $m < -1$ and $M > 1$. Applying (3.1) to (3.4) and (3.5), we then get

$$d(M-1) \leqslant (M - m) \int_N |J| \tag{3.6}$$

and

$$d(m+1) \geqslant (m - M) \int_N |J| \tag{3.7}$$

From (3.7) and (3.2) we obtain

$$-m \leqslant \frac{d + M \int_N |J|}{d - \int_N |J|}$$

Substituting this into (3.6) we get

$$M\left(d-\int_N |J| - \frac{(\int_N |J|)^2}{d-\int_N |J|}\right) \leqslant d + \frac{d\int_N |J|}{d-\int_N |J|}$$

Thus, because of (3.2) we have

$$M \leqslant \frac{d}{d-2\int_N |J|}$$

and

$$m \geqslant \frac{-d}{d-2\int_N |J|}$$

Finally, if $M \leqslant 1$ and $m < -1$, then (3.7) and (3.2) imply that

$$m \geqslant \frac{-d-\int_N |J|}{d-\int_N |J|} \geqslant \frac{-d}{d-2\int_N |J|}$$

A similar argument shows that $M > 1$ and $m \geqslant -1$ again implies

$$M \leqslant \frac{d}{d-2\int_N |J|}$$

which completes the proof. ∎

Remark 3.2. Note that if $J \geqslant 0$ and u is a nonconstant solution of (1.1), then (3.3) at first implies

$$-1 \leqslant u(x) \leqslant 1 \qquad \text{for all} \quad x \in R^n$$

but one then can easily see from (1.1) that

$$-1 < u(x) < 1 \qquad \text{for all} \quad x \in R^n$$

actually holds for such nonconstant solutions.

Obviously,

$$\frac{-d}{d-2\int_N |J|} \int_{R^n} |J| \leqslant J * u \leqslant \frac{d}{d-2\int_N |J\,|} \int_{R^n} |J| \equiv b$$

Let us assume that λ is large enough that

$$|1+\lambda f'(u)| > \int_{R^n} |J| \qquad \text{whenever} \quad |u+\lambda f(u)| \leqslant b \tag{3.8}$$

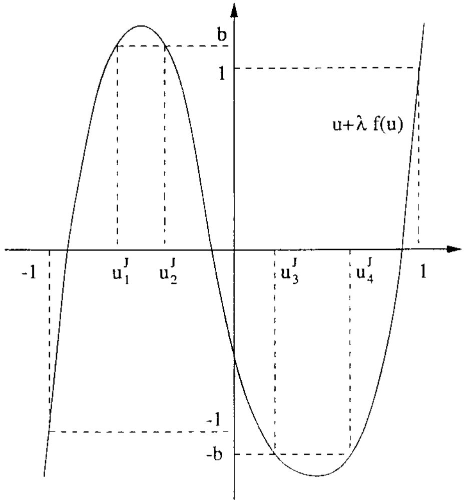

Fig. 1. $u+\lambda f(u)$.

Let u_1^J, u_2^J be the two zeros of $u+\lambda f(u)-b$ such that $-1<u_1^J<u_2^J<1$ and u_3^J, u_4^J the two zeros of $u+\lambda f(u)+b$ such that $-1<u_3^J<u_4^J<1$ (see Fig. 1).

Define

$$I_1^J \equiv \left[\frac{-d}{d-2\int_N |J|}, u_1^J\right],$$

$$I_2^J \equiv [u_2^J, u_3^J],$$

$$I_3^J \equiv \left[u_4^J, \frac{d}{d-2\int_N |J|}\right]$$

From the definition of b we immediately have

Proposition 3.3. Assume (3.2) and (3.8) hold. Then any $L^\infty(R^n)$ solution u to (1.1), satisfies

$$u(x) \in I_1^J \cup I_2^J \cup I_3^J \qquad \text{for all} \quad x \in R^n \tag{3.9}$$

We now state our theorem.

Theorem 3.4 (Existence). Assume that f satisfies (3.8). All solutions of (1.1) are characterized as follows.

Let M_1 and M_2 be any two disjoint measurable sets. Then there exists a unique solution $\hat{u}_J$ to (1.1), such that $\hat{u}_J(x) \in I_1^J$ for $x \in M_1$, $\hat{u}_J(x) \in I_2^J$ for

$x \in M_2$ and $\hat{u}_J(x) \in I_3^J$ for $x \in (M_1 \cup M_2)^c$. Moreover, $\hat{u}_J$ is C^m on int M_1, int M_2 and int$(M_1 \cup M_2)^c$.

Proof. Let M_1 and M_2 be any two disjoint measurable sets. Define

$$F(u, \alpha) = \alpha(J * u - u) - f(u)$$

and

$$u_0(x) = \begin{cases} -1, & x \in M_1, \\ a, & x \in M_2, \\ 1, & x \in (M_1 \cup M_2)^c \end{cases} \tag{3.10}$$

Note that $F(u_0, 0) = 0$. Let $L_0 \equiv (\partial F/\partial u)(u_0, 0)$. Then, since

$$L_0 v = -f'(u_0)\, v \tag{3.11}$$

L_0 is invertible in $L^\infty(R^n)$, thus, by the Implicit Function Theorem, there exists some $\alpha_0 > 0$, such that there exists a locally unique solution u_α of $F(u, \alpha) = 0$ for $|\alpha| \leqslant \alpha_0$.

Consider the family of equations parameterized by $\lambda \in [\lambda_*, 1/\alpha_0]$:

$$G(u, \lambda) \equiv J * u - u - \lambda f(u) = 0 \tag{3.12}$$

where λ_* is the infimum of λ's for which (3.8) holds, and without loss of generality $\lambda_* < 1/\alpha_0$. Clearly, G is C^1 on $L^\infty(R^n) \times R^1$. When $\lambda = 1/\alpha_0$, (3.12) has the solution u_{α_0}. The Implicit Function Theorem is now used to obtain the same conclusion for all $\lambda \in [\lambda_*, 1/\alpha_0]$:

Let u_λ be a solution of $G(u, \lambda) = 0$ and L_λ be the linear operator defined in $L^\infty(R^n)$ by

$$L_\lambda v = J * v - [1 + \lambda f'(u_\lambda)]\, v \tag{3.13}$$

Note that $L_\lambda \equiv (\partial G/\partial u)(u_\lambda, \lambda)$.

Let $\lambda_0 \in (\lambda_*, 1/\alpha_0]$ be such that a solution, u_{λ_0}, exists to the equation $G(u, \lambda_0) = 0$. First, we show that there exists $\eta > 0$ such that for $\lambda \in (\lambda_0 - \eta, \lambda_0]$, (3.12) has a solution.

By the Implicit Function Theorem, it suffices to show that L_{λ_0} is invertible. (3.13) can be rewritten as

$$L_{\lambda_0} v = [1 + \lambda f'(u_{\lambda_0})] \left[\frac{1}{1 + \lambda f'(u_{\lambda_0})} J * v - v \right] \tag{3.14}$$

Since from (3.9) we *a priori* know that $|1+\lambda f'(u_{\lambda_0})| > \|J\|_{L^1(R^n)}$, it is easily seen that L_{λ_0} is invertible.

To show that we can continue the solution branch to $\lambda \in [\lambda_*, 1/\alpha_0]$, we argue by contradiction. Suppose that there is some $\bar{\lambda} \geqslant \lambda_*$ such that a solution exists for $\lambda \in (\bar{\lambda}, 1/\alpha_0]$, but not for $\lambda = \bar{\lambda}$. Note that for any $x_1, x_2 \in M_1$ we have

$$|u_\lambda(x_1) - u_\lambda(x_2)| \leqslant \frac{|DJ * u_\lambda|_\infty}{\min_{x \in M_1} \{|1+\lambda f'(u_\lambda(x))|\}} \leqslant \text{const}$$

where DJ is the (weak) derivative of J. Similar inequalities hold for any $x_1, x_2 \in M_2$ or any $x_1, x_2 \in (M_1 \cup M_2)^c$. This implies that the family of solutions u_λ, $\lambda \in (\bar{\lambda}, 1/\alpha_0]$, is equicontinuous on M_1, M_2 and $(M_1 \cup M_2)^c$. Thus, by the Arzela–Ascoli Theorem, we can pass to the limit along a sequence $\lambda_n \to \bar{\lambda}$ to obtain a solution of $G(u, \bar{\lambda}) = 0$.

This completes the existence proof. Note that we have existence for $\lambda = \lambda_*$, where (3.8) may fail.

To show uniqueness when (3.8) holds, assume that there are two distinct solutions u_1 and u_2 of (1.1), such that $u_1(x), u_2(x) \in I_1^J$ for $x \in M_1$, $u_1(x), u_2(x) \in I_2^J$ for $x \in M_2$ and $u_1(x), u_2(x) \in I_3^J$ for $x \in (M_1 \cup M_2)^c$. Then

$$|u_1 - u_2|_\infty \leqslant |g_i^{-1}(J * u_1) - g_i^{-1}(J * u_2)|_\infty \leqslant k\, |u_1 - u_2|_\infty$$

where $g(u) = u + \lambda f(u)$ and g_i^{-1}, $i = 1, 2, 3$, is defined to be one of the three branches of g^{-1} and $k < 1$ by (3.8). Thus $u_1 \equiv u_2$ a.e.

Finally, regularity follows from a similar argument as was used in Theorem 2.1. ∎

We now provide a stability theorem for the solutions constructed in Theorem 3.4.

Theorem 3.5 (Stability). Let $u(x)$ be a solution of (1.1). Then

1. if $u(x) \in I_1^J \cup I_3^J$ for all $x \in R^n$, u is (locally) exponentially asymptotically stable in the $L^\infty(R^n)$ norm.

2. if $u(x) \in I_2^J$ for $x \in M_2$, where M_2 is a measurable subset of R^n, such that $|M_2| > 0$, then u is unstable in the $L^\infty(R^n)$ norm.

Proof. First, we investigate $\sigma(L_\lambda)$, the spectrum of L_λ acting on L^∞, defined in (3.13). Note that $(L_\lambda - \mu)\, v = J * v - [1 + \lambda f'(u) + \mu]\, v$ is invertible for

$$\mu \in \bigcap_{r \in S} \{z \colon |z + r| > \|J\|_{L^1(R^n)}\}$$

where $S \equiv \{1+\lambda f'(u(x))\colon x \in R^n\}$, since

$$(L_\lambda - \mu)\, v = [1+\lambda f'(u)+\mu]\left[\frac{1}{1+\lambda f'(u)+\mu}\, J * v - v\right]$$

Thus,

$$\sigma(L_\lambda) \subset \bigcup_{r \in S} \{z\colon |z+r| \leqslant \|J\|_{L^1(R^n)}\} \tag{3.15}$$

If $u(x) \in I_1^J \cup I_3^J$ for all $x \in R^n$, then $\sigma(L_\lambda)$ lies in the left-half plane, thus, by ref. 4, u is (locally) asymptotically stable.

Assume on the other hand, that $u(x) \in I_2^J$ for $x \in M_2$, where M_2 is a measurable subset of R^n, such that $|M_2| > 0$. From the construction in Theorem 3.4, this solution is continued from u_0 given by (3.10). Note that

$$\sigma(L_0) \subset \{-f'(-1),\ -f'(a),\ -f'(1)\}$$

where L_0 is given by (3.11). Each of the points in $\sigma(L_0)$ is an eigenvalue of infinite multiplicity. Since our continuation is a C^1 deformation, by (3.15), $\sigma(L_\lambda)$ does not intersect the imaginary axis and $\sigma(L_\lambda)$ contains values in the right-half plane. Thus, from ref. 4 we conclude that u is unstable. ∎

Any solution $u(x)$ constructed in Theorem 3.4 is such that $|1+\lambda f'(u(x))| > 1$ for all $x \in R^n$. We now show that the instability criterion in Theorem 3.5 can be relaxed to accommodate other solutions to (1.1) with general non-monotone g.

Theorem 3.6 (Instability). Assume that $J \geqslant 0$ and λ is such that $1+\lambda f'(u) < 0$ on some interval. Let M_2 be a measurable subset of R^n, such that $|M_2| > 0$, and let u be a solution to (1.1) such that $1+\lambda f'(u(x)) < 0$ for $x \in M_2$. Then u is unstable in the $L^\infty(R^n)$ norm.

Proof. For some positive numbers ε and β to be chosen later, let

$$\underline{u}(x,t) \equiv \begin{cases} u(x) + \varepsilon e^{\beta t} & \text{for} \quad x \in M_2, \\ u(x) & \text{for} \quad x \in M_2^c \end{cases}$$

Define $Nv \equiv v_t - (J * v - v - \lambda f(v))$. It is easily seen that

$$N\underline{u}(x,t) = \begin{cases} \varepsilon\beta e^{\beta t} - \Big(\Big(\int_{M_2} J(x-y)(u(y)+\varepsilon e^{\beta t})\,dy \\ \quad + \int_{M_2^c} J(x-y)\,u(y)\,dy - u(x) - \varepsilon e^{\beta t} - \lambda f(u(x)+\varepsilon e^{\beta t}) \Big) & \text{for } x \in M_2 \\ -\Big(\Big(\int_{M_2} J(x-y)(u(y)+\varepsilon e^{\beta t})\,dy \\ \quad + \int_{M_2^c} J(x-y)\,u(y)\,dy - u(x) - \lambda f(u(x)) \Big) & \text{for } x \in M_2^c \end{cases}$$

$$= \begin{cases} \varepsilon\beta e^{\beta t} + \lambda f(u(x)+\varepsilon e^{\beta t}) - \lambda f(u(x)) + \varepsilon e^{\beta t} \int_{M_2^c} J(x-y)\,dy & \text{for } x \in M_2 \\ -\varepsilon e^{\beta t} \int_{M_2} J(x-y)\,dy & \text{for } x \in M_2^c \end{cases}$$

By Taylor's expansion, $|\lambda f(u+\varepsilon e^{\beta t}) - \lambda f(u) - \lambda f'(u)\,\varepsilon e^{\beta t}| \leqslant C(\varepsilon e^{\beta t})^2$ for some $C > 0$. Thus,

$$N\underline{u}(x,t) \leqslant \begin{cases} \varepsilon e^{\beta t}(\beta + \lambda f'(u(x)) + C\varepsilon e^{\beta t} + \int_{M_2^c} J(x-y)\,dy) & \text{for } x \in M_2, \\ 0 & \text{for } x \in M_2^c \end{cases}$$

$$\leqslant \begin{cases} \varepsilon e^{\beta t}(\beta + 1 + \lambda f'(u(x)) + C\varepsilon e^{\beta t}) & \text{for } x \in M_2, \\ 0 & \text{for } x \in M_2^c \end{cases}$$

Since $1 + \lambda f'(u(x)) < 0$ for $x \in M_2$, we can choose ε, $\beta > 0$ such that

$$N\underline{u} \leqslant 0 \qquad \text{for} \quad 0 < t \leqslant t_0 \tag{3.16}$$

where t_0 is such that

$$C\varepsilon e^{\beta t_0} = \inf_{x \in M_2} \{-1 - \lambda f'(u(x)) - \beta\}$$

Thus, $\underline{u}$ is a subsolution of the Cauchy problem for (1.2) up till time t_0. But it is easily computed that the t-independent function $\underline{u}(x, t_0)$ satisfies

$$N\underline{u}(x,t_0) \leqslant \begin{cases} \varepsilon e^{\beta t_0}(\lambda f'(u(x)) + C\varepsilon e^{\beta t_0} + \int_{M_2^c} J(x-y)\,dy) & \text{for } x \in M_2, \\ -\int_{M_2} J(x-y)\,\varepsilon e^{\beta t_0} & \text{for } x \in M_2^c \end{cases}$$

$$\leqslant \begin{cases} \varepsilon e^{\beta t_0}(-\beta) & \text{for } x \in M_2, \\ 0 & \text{for } x \in M_2^c \end{cases}$$

so $N\underline{u} \leqslant 0$ and $\underline{u}$ is also a subsolution. By the comparison principle for (1.2) this means that the solution $u(x, t)$ of the Cauchy problem with initial data $\underline{u}(x, 0)$ is such that

$$\|u(\cdot, t) - u(\cdot)\|_\infty \geqslant \varepsilon e^{\beta t_0} = \inf_{x \in M_2} \left\{ \frac{-1 - \lambda f'(u(x)) - \beta}{C} \right\} \qquad \text{for all} \quad t \geqslant t_0$$

Since the expression on the right is independent of ε, instability follows.

Remark 3.7. The assumption $J \geqslant 0$ is necessary here, since we are using the comparison principle for (1.2), which is not applicable when J changes sign.

4. GLOBAL STABILITY FOR λ LARGE AND $n = 1$

For $n = 1$, the domains of attraction of a class of asymptotically stable solutions (constructed in the previous section) can be characterized as follows.

Assume that $J \geqslant 0$ and $\int_{R^1} |x|\, J(x)\, dx < \infty$. Then u_1^J, u_2^J, u_3^J, u_4^J, I_1^J, I_2^J, I_3^J defined as in the previous section do not depend on J. Thus in the following, we omit the superscript J.

Theorem 4.1 (Global Stability). Let $\hat{u}$ be an asymptotically stable solution of (1.1), such that $\hat{u}(x) \in I_1$ for $x \in M$ and $\hat{u}(x) \in I_3$ for $x \in M^c$, for some measurable M with M and M^c both having positive measure. Assume that $\sup\{|z|: z \in T\} < \infty$, where $T \equiv \bar{M} \cap \overline{M^c}$ is the set of jump discontinuities of $\hat{u}$. Consider the Cauchy problem: (1.2) with the initial condition $u(x, 0) = u_0(x)$. Assume that $-1 \leqslant u_0(x) \leqslant 1$,

$$\begin{cases} u_0(x) \in [-1, u_2] & \text{for} \quad x \in M, \\ u_0(x) \in [u_3, 1] & \text{for} \quad x \in M^c \end{cases} \tag{4.1}$$

and u_0 is continuous on M and M^c. Then the solution $u(x, t)$ of the Cauchy problem converges exponentially to $\hat{u}(x)$.

Proof. By some results in ref. 3, there are two monotone solutions of (1.1), U_1 and U_2 such that $U_1(-\infty) = -1$, $U_1(+\infty) = 1$, $U_2(-\infty) = 1$, $U_2(+\infty) = -1$ and $U_1(x), U_2(x) \in I_1 \cup I_3$. These solutions have the nice property that the solution $u(x, t)$ of the Cauchy problem (1.2) with an initial condition $u(x, 0) = u_0(x)$ is "squeezed" between a translate of U_1 (or U_2) and -1 (or 1). To be more precise, for some $\mu > 0$, depending upon u_0, there exist constants $x_1, x_2 > 0$ and $\delta > 0$, such that the following four implications hold for all $x \in R^1$ and $t > 0$:

$$
\begin{aligned}
u(x, t) &< U_1(x - x_1) + \mu e^{-\delta t} && \text{if} \quad \limsup_{x \to -\infty} u_0(x) < a, \\
u(x, t) &> U_2(x - x_1) - \mu e^{-\delta t} && \text{if} \quad \liminf_{x \to -\infty} u_0(x) > a, \\
u(x, t) &< U_2(x - x_2) + \mu e^{-\delta t} && \text{if} \quad \limsup_{x \to +\infty} u_0(x) < a, \\
u(x, t) &> U_1(x - x_2) - \mu e^{-\delta t} && \text{if} \quad \liminf_{x \to +\infty} u_0(x) > a
\end{aligned}
\tag{4.2}
$$

Note that by the assumption that $\sup\{|z|: z \in T\} < \infty$, where T is the set of jump discontinuities of u, exactly two of the above inequalities are satisfied.

We show that there is a $t_1 > 0$ such that

$$u(x, t) \in [-1, u_1] \cup [u_4, 1] \qquad \text{for} \quad t > t_1 \tag{4.3}$$

It suffices to show that there is some $\varepsilon > 0$, such that

$$
\begin{cases}
u_t(x, t) < -\varepsilon & \text{for all} \quad (x, t) \text{ such that } u(x, t) \in [u_1, u_2], \\
u_t(x, t) > \varepsilon & \text{for all} \quad (x, t) \text{ such that } u(x, t) \in [u_3, u_4]
\end{cases}
\tag{4.4}
$$

(4.3) will then follow by setting

$$t_1 \equiv \max \left\{ \frac{u_2 - u_1}{\varepsilon}, \frac{u_4 - u_3}{\varepsilon} \right\}$$

To see, say, the first of the inequalities in (4.4), we argue as follows:

Fix $t_0 > 0$, let $A = \{x: u(x, t_0) = 1\}$, and with K denoting the support of J, let $K_x = \{y: x - y \in K\}$. It is easy to see that if $A \neq \emptyset$ then there is a point $x_0 \in A$ such that $|K_{x_0} \cap A^c| > 0$. Hence,

$$J * u(x_0, t_0) - g(u(x_0, t_0)) = \int_{R^1} J(x_0 - y)(u(y, t_0) - 1)\, dy < 0$$

since $u(y, t_0) < 1$ on A^c. This implies that $u_t(x_0, t_0) < 0$ while $u(x_0, t_0) = 1$ and $u(x_0, t) \leqslant 1$ for $0 \leqslant t \leqslant t_0$, a contradiction. Thus, A is empty and $u(x, t) < 1$ for all $t > 0$ and $x \in R^1$. Similarly, $u(x, t) > -1$ for all $t > 0$ and $x \in R^1$.

Again, fix $t_0 > 0$ and consider x_0 such that $u(x_0, t_0) \in [u_1, u_2]$. Note that the set $D(t_0)$ of such points x_0 is contained in a compact interval for a suitable choice of t_0 by (4.2). Furthermore, this interval does not increase for $t > t_0$. Since $g(u(x_0, t_0)) \geqslant 1$ and $u(y, t_0) < 1$ for all $y \in R^1$, we have

$$J * u(x_0, t_0) - g(u(x_0, t_0)) \leqslant \int_{R^1} J(x_0 - y)(u(y, t_0) - 1)\, dy < 0$$

This is uniform in $t \geqslant t_0$ since (4.2) prevents $u(y, t)$ approaching 1 a.e. on K_{x_0}. That is, there is a constant $\varepsilon_0 > 0$ such that $u_t(x_0, t) \leqslant -\varepsilon_0$ for $t \geqslant t_0$. This ε_0 depends continuously upon x_0 lying in a compact set, and hence we can choose $\varepsilon > 0$ uniformly, as claimed.

Since $1 + \lambda f'(u(x, t)) > 1$ for all $x \in R^1$ and $t > t_1$, the modulus of continuity of $u(x, t)$ in x is bounded on any $[a, b] \cap M$ and $[a, b] \cap M^c$, uniformly in $t \geqslant t_1$ (see refs. 3 or 12 for easy proofs). This, together with the Arzela–Ascoli Theorem and (4.2) implies that for any sequence $t'_n \to \infty$ there exists a subsequence $t_n \to \infty$ such that

$$u(x, t_n) \to \text{ some } u_\infty(x) \text{ in } L^\infty(R^1) \text{ as } t_n \to \infty \tag{4.5}$$

Note that $u_\infty(\pm\infty) = -1$ or 1.

It now suffices to show that u_∞ is a solution of (1.1) (from uniqueness, it will then follow that $u_\infty \equiv \hat{u}$). The idea follows the lines of ref. 3 and we only outline the proof here.

Let η be a C^∞ function defined on $[0, \infty)$ such that $\eta(x) = 1$ for $x \in [0, 1/2]$ and $= 0$ for $x \geqslant 1$. Let

$$w(x, t) \equiv \begin{cases} u(x, t) & \text{for } \ |x| \leqslant t, \\ u(x, t)\, \eta(x - t) + (u_\infty(+\infty) - \eta(x - t)) & \text{for } \ x \geqslant t, \\ u(x, t)\, \eta(-x - t) - (u_\infty(-\infty) - \eta(-x - t)) & \text{for } \ x \leqslant -t. \end{cases}$$

Then $w(x, t) \equiv u_\infty(+\infty)$ for $x \geqslant t + 1$ and $w(x, t) \equiv u_\infty(-\infty)$ for $x \leqslant -t - 1$. Define an energy functional associated with (1.2) by

$$V(t) = \tfrac{1}{2} \int_{R^1} [(J * w - w)\, w - \lambda F(w) + \lambda H(x)\, F(u_\infty(\infty))]\, dx$$

where $H(x)$ is the Heaviside step function and $F(u) \equiv \int_{u_\infty(-\infty)}^{u} f(s)\, ds$. V clearly converges, because of the truncation. Using an argument similar to one in ref. 3, it can be shown that $V(t)$ is bounded independently in $t \geqslant 0$. It is here (and only here) that the assumption $\int_{R^1} |x|\, J(x)\, dx < \infty$ is necessary. Next, we differentiate $V(t)$ to obtain

$$\dot{V}(t) = \int_{R^1} (J * w - w - \lambda f(w))\, w_t\, dx$$

Let

$$Q(t) \equiv \int_{R^1} (J * w - w - \lambda f(w))^2\, dx$$

It can then be shown that

$$\lim_{t \to \infty} (\dot{V}(t) - Q(t)) = 0 \tag{4.6}$$

(see ref. 3 for details). Since $Q(t) \geqslant 0$ it follows that $\liminf_{t \to \infty} \dot{V}(t) \geqslant 0$. We thus deduce the existence of a sequence $\{t'_n\}$, with $t'_n \to \infty$ such that

$$\lim_{n \to \infty} \dot{V}(t'_n) = 0$$

(for otherwise $\liminf_{t \to \infty} \dot{V}(t) > 0$ implying that $V(t) \to \infty$, in contradiction with the fact that $V(t)$ is bounded). By (4.6), Fatou's Lemma, (4.5), and by passing to the limit along the subsequence $\{t_n\}$ of $\{t'_n\}$, we have

$$\int_{R^1} (J * u_\infty - u_\infty - \lambda f(u_\infty))^2 \, dx = 0$$

thus u_∞ is a solution of (1.1).

The convergence is uniform, and to finish the proof we apply Theorem 3.5(1), which says that if $u(x, t)$ lies in a ε-neighborhood of u_∞ for some t_n, then $u(x, t) \to u_\infty(x)$ for $t \to \infty$.

Remark 4.2. Note that in particular, this results provides another proof of existence of stable solutions.

5. ASYMPTOTIC BEHAVIOR OF MONOTONE STATIONARY WAVES

Let $n = 1$ and consider the increasing solution $\bar{u}(x)$ to (1.1), constructed in Theorem 3.4 and ref. 3, having only one jump discontinuity. Assume that the jump occurs at $x = 0$. It was shown in ref. 3 that $\bar{u}' > 0$.

Theorem 5.1. Let $\bar{u}$ be as above. Then

$$\limsup_{x \to \pm\infty} \frac{J(x)}{\bar{u}'(x)} < \infty$$

Proof. $\bar{u}$ satisfies $J * u = g(u)$. Differentiating this equation gives

$$g'(\bar{u}(x))\, \bar{u}'(x) = J(x)(\bar{u}(0+) - \bar{u}(0-)) + (J * \bar{u}')(x)$$

Dividing this equation by $\bar{u}'(x)$ and observing that $(J * \bar{u}')(x) \geqslant 0$ for all $x \in R^1$ then yields

$$\limsup_{x \to \pm\infty} \frac{J(x)}{\bar{u}'(x)} < \infty$$

Remark 5.2. This shows that the convergence of $\bar{u}$ to ± 1 is not faster than that of $\int J(x)\,dx$, in particular it need not be exponential.

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